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Multiplicity difference between heavy- and light-quark jets revisited

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Abstract. The perturbative QCD approach to multiparticle production predicts a characteristic suppression of particle multiplicity in a heavy-quark jet as compared to a light-quark jet. In the modified leading logarithmic approximation (MLLA) the multiplicity difference $\delta_{Q\ell}$ between heavy- and light-quark jets is derived in terms of a few other experimentally measured quantities. The earlier prediction for b-quarks needs revision in the light of new experimental results and the improvement in the understanding of the experimental data. We now find $\delta_{b\ell} = 4.4 \pm 0.4$. The updated MLLA results on $\delta_{b\ell}$ and $\delta_{c\ell}$ are compared with the present data from e^+e^- annihilation. Their expected energy independence is confirmed within the energy range between 29 and 200 GeV; the absolute values are now in better agreement with experiment than in the previous analysis, and the remaining difference can be attributed largely to next-to-MLLA contributions, an important subset of which are identified and evaluated.

1 Introduction

Since the early days of QCD, heavy-quark physics has been one of the primary testing grounds for many aspects of the theory. In the last years a wealth of new important results on the profile of jets initiated by heavy quarks $Q(b,c)$ has been reported by the experimental collaborations at LEP, SLC, Tevatron and HERA. Future progress is expected from the measurements at the LHC and a future linear e^+e^- collider. These studies are important for the tests of the basic concepts of the QCD description of multiparticle production and also for the studies of new physics.

Multiple hadron production in hard processes is derived from the QCD parton cascade processes which are dominated by gluon bremsstrahlung. An essential difference in the structure of the energetic heavy- and light-quark jets ($\ell \equiv q = u,d,s$) results from the dynamical restriction on the phase space of primary gluon radiation in the heavy-quark case: the gluon radiation off an energetic quark Q with mass M and energy $E_Q \gg M$ is suppressed inside the forward angular cone with an opening angle $\Theta_0 = M/E_Q$, the so-called dead cone phenomenon [1,2]. This is in close analogy with QED where the photon radiation is suppressed at small angles with respect to a primary charged massive particle. The suppression of the energetic gluon emission at low momentum transfer $k_\perp$ results, in turn, in the decrease of the heavy-quark energy losses. This provides a pQCD explanation of the leading particle effect [3,4] which is clearly seen experimentally in the $b\bar{b}$ and $c\bar{c}$ events in e^+e^- annihilation [5]; for recent reviews, see [6].

For a long time, there has been no clear explicit experimental visualisation of the dead cone. Only recently, preliminary DELPHI results have been reported [7] which show the expected depletion of small angle particle production in b-jets with respect to the heavy hadron direction. Further detailed studies of the dead cone effect in different processes are needed. Some new results may come from the current analysis of the structure of the c-quark jets, produced in photon gluon fusion in deep inelastic scattering at HERA.

It is worthwhile to mention that the difference in the radiation from massive and massless quarks should also manifest itself in the QCD medium via suppression of the *medium-induced radiative energy loss* of heavy quarks propagating in a strongly interacting matter; see, for example, [8–11] and references therein.

Studies of heavy-quark jets are also important in the investigation of the properties of known or new heavy objects. For example, a detailed knowledge of the b-jet profile is needed for the analysis of the final state in the $t\bar{t}$ production processes. Various aspects of studying new physics, in particular, of the structure of the Higgs sector at the

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LHC and at a future linear collider, would benefit from the detailed understanding of the b-initiated jets; see for example [12,13].

The dead cone phenomenon leads to essential differences in the profiles of the light- and heavy-quark-initiated jets. According to the concept of "local parton hadron duality" (LPHD) [14], the dead cone suppression of gluon radiation should result in the characteristic differences in "companion" spectra and multiplicities of primary light hadrons in these jets [1,2,15].

In particular, as a direct consequence of the LPHD scenario, the difference of companion multiplicities N^h of light hadrons in the heavy-quark and light-quark jets at the same jet energy E_{jet} should be *energy independent* (up to a power correction $\mathcal{O}\left(M^2/E_{\text{jet}}^2\right)$), i.e. in e^+e^- annihilation at CMS energy $W = 2E_{\text{jet}}$ one obtains the QCD prediction [6,15]

$$N^h_{q\bar{q}}(W) - N^h_{Q\overline{Q}}(W) = \text{const}(W). \tag{1}$$

The corresponding constant is different for c- and b-quarks and depends on the type of light hadrons h under study.

This prediction is in marked contrast with the expectation of the so-called naïve model [16], which relates the multiplicities in light- and heavy-quark events based on the idea of the *reduction of the energy scale*,

$$N^h_{Q\overline{Q}}(W) = N^h_{q\bar{q}}\left((1-\langle x_Q\rangle)W\right); \tag{2}$$
$$\langle x_Q\rangle = \frac{2\langle E_Q\rangle}{W}, \quad 1-\langle x_Q\rangle = \mathcal{O}\left(\alpha_s(W)\right).$$

In this case the difference of q- and Q-induced multiplicities would *grow* gradually with W as

$$N^h_{q\bar{q}}(W) - N^h_{Q\overline{Q}}(W) \propto \sqrt{\alpha_s}\ln\frac{1}{\alpha_s}\cdot N^h_{q\bar{q}}(W). \tag{3}$$

In this paper we focus on the analysis of the current experimental status of the difference of the average charged multiplicities $\delta_{b\ell}$ of events containing b- and light quarks in e^+e^- annihilation in the available energy range. The situation with charmed quarks is considered as well. The main emphasis is on the comparison between the reanalysed data and the expectations based on the MLLA, in an extension of previous analyses [6,15]. In addition, we discuss the size of the next-to-MLLA contributions.

2 Theoretical analysis

Within the LPHD framework, the multiplicity of light hadrons in e^+e^- annihilation events is proportional to that of bremsstrahlung partons. To predict the QCD yield of light particles accompanying $Q\overline{Q}$ production we have first to address the question on how the development of the parton cascade initiated by a heavy quark Q depends on the quark mass M.

2.1 Structure of QCD cascades in $e^+e^- \to Q\overline{Q} + \ldots$

As is well known, in the case of a light-quark jet the structure of the parton branching of the primary gluon g_1 with energy ω_1 (energy spectra, multiplicities of secondaries) is determined by the parameter

$$\kappa_q = 4\omega_1^2\sin^2\frac{\Theta_1}{2}, \tag{4}$$

where Θ_1 is the angle between the gluon and the energetic quark. (The expression (4) is written in such a way as to account for the next-to-leading correction due to large angle soft gluon emission, up to the full jet opening angle $\Theta_1 = \pi$; see for example [17,18].) For $\Theta_1 \ll 1$ this parameter reduces to the gluon transverse momentum, $k_t^2 \simeq (\omega_1\Theta_1)^2$. The appearance of this scale is a consequence of colour coherence in multiplication of soft gluons which dominate the QCD cascades. This destructive coherence results in the *angular ordering* (AO) of successive parton branchings [19].

The corresponding parameter for a jet initiated by a *heavy* quark with energy E_Q and the mass M reads

$$\kappa_Q = \omega_1^2\left[\left(2\sin\frac{\Theta_1}{2}\right)^2 + \Theta_0^2\right]; \qquad \Theta_0 \equiv \frac{M}{E_Q}. \tag{5}$$

Note that the same quantity κ_Q determines the scale of the running coupling in gluon emission off the massive quark.[1]

The modification of the angular parameter in (5) caused by the heavy-quark mass has a transparent physical interpretation.[2] Consider radiation of a secondary gluon g_2 with energy $\omega_2 \ll \omega_1$ at angle Θ_{21} relative to the primary gluon g_1. Normally, in the "disordered" angular kinematics, $\Theta_{21} > \Theta_1$, the destructive interference between the emission amplitudes of g_2 off the quark and g_1 *cancels* the independent radiation $g_1 \to g_2$, thus enforcing

$$\Theta_{21} \leq \Theta_1. \tag{6}$$

In the meantime, in the massive quark case the interference contribution enters the game only when the angle Θ_2 of g_2 with respect to the *quark* is larger than the dead cone, $\Theta_2 > \Theta_0$. Therefore, the cancellation leading to the AO condition (6) does not occur when the gluon g_1 is radiated *inside* the dead cone, $\Theta_1 < \Theta_0$, and the jet evolution parameter (5) *freezes* in the $\Theta_1 \to 0$ limit.

In physical terms what happens is the *loss of coherence* between Q and g_1 as emitters of the soft gluon g_2 due to accumulated longitudinal separation $\Delta z > \lambda_{||}^{(2)} \approx \omega_2^{-1}$ between the massive and massless charges ($v_Q \approx 1-\Theta_0^2/2 < 1$, $v_1 = 1$). Indeed, during the formation time of the secondary radiation, $t_f^{(2)} \sim (\omega_2\Theta_{21}^2)^{-1}$, the two sources – the quark and the gluon g_1 – separate in the longitudinal direction by

$$\Delta z \sim t_f^{(2)}\left|v_Q - c\cos\Theta_1\right| \simeq \lambda_{||}^{(2)}\cdot\frac{\Theta_1^2+\Theta_0^2}{\Theta_{21}^2}. \tag{7}$$

[1] A detailed analysis of the running coupling argument in the massive quark case can be found in the appendices to [3]; see also [6,20].

[2] This argument is based on the discussion of two of the authors (YLD,VAK) with Troyan in the early 90's.

It is the last factor that determines whether an interference is essential or not. When this ratio is larger than 1, the quark Q and gluon g_1 are separated enough for g_2 to be able to resolve the two emitters as independent colour charges. In these circumstances g_1 acts as an independent source of the next generation bremsstrahlung quanta. Otherwise, no additional particles triggered by g_1 emerge on top of the yield determined by the *quark* charge (which equals the total colour charge of the $Q + g_1$ system).

In the massless quark case ($\Theta_0 \equiv 0$) this consideration reproduces the standard AO prescription (6). In the massive quark case, the separation (*incoherence*) condition $\Theta_{21}^2 \leq (\Theta_1^2 + \Theta_0^2)$ results in (5) as the proper evolution parameter for the gluon subjet.

The modification (5) may look superfluous since the soft gluon radiation inside the dead cone, $\Theta_1 \ll \Theta_0$, is suppressed. In spite of this, it is essential for keeping track of the next-to-leading order (MLLA) corrections in accompanying multiplicities. In Appendix A we recall the structure of the exact matrix element for gluon radiation off a heavy $Q\overline{Q}$ pair and show how the parameter (5) naturally appears in the problem.

2.2 MLLA prediction for accompanying multiplicity and its accuracy

The light charged hadron multiplicity in heavy-quark e^+e^- annihilation events at CMS energy W can be represented as

$$N_Q^{\rm ch}(W) \equiv N_{e^+e^-\to Q\overline{Q}}^{\rm ch}(W) = N_{Q\overline{Q}}^{\rm ch}(W) + n_Q^{\rm dc}, \quad (8)$$

where $N_Q^{\rm ch}$ is the charged multiplicity of e^+e^- events containing a heavy quark Q; $N_{Q\overline{Q}}^{\rm ch}(W)$ is the charged multiplicity of light hadrons accompanying the heavy-quark production (excluding decay products of Q-flavoured hadrons) and $n_Q^{\rm dc}$ stands for the constant charged decay multiplicity of the two leading heavy hadrons ($n_b^{\rm dc} = 11.0 \pm 0.2$ for b-quarks, $n_c^{\rm dc} = 5.2 \pm 0.3$ for c-quarks; see for example [15] for details of a previous evaluation). As shown in Appendix A, at $W = 2E_Q \gg M \gg \Lambda_{\rm QCD}$ the companion multiplicity $N_{Q\overline{Q}}(W)$ can be related to the particle yield in the *light-quark* events $e^+e^- \to q\bar{q}$ ($q = u, d, s$) as [6,15]

$$N_{q\bar{q}}(W) - N_{Q\overline{Q}}(W) = N_{q\bar{q}}(\sqrt{e}M) \cdot [1 + \mathcal{O}(\alpha_{\rm s}(M))], \quad (9)$$

where we approximately expressed the *difference* between the light- and heavy-quark generated multiplicities in terms of the light-quark event multiplicity at reduced (W independent) CMS energy $W_0 = \sqrt{e}M$, $e = \exp(1)$.

Concerning the *accuracy* of (9), there are two separate issues one has to address, namely the following.
(1) The accuracy of the statement of the *constancy* of the LHS of (9), and
(2) the accuracy to which this difference can be quantitatively predicted by means of pQCD (the RHS).

Left-hand side

Answering the first question, it turns out to be insufficient to compare particle multiplicities in a given order of perturbation theory. Indeed, within the next-to-leading accuracy (MLLA), for example, one takes into consideration ("exponentiated") $\sqrt{\alpha_{\rm s}} + \alpha_{\rm s}$ effects in the anomalous dimension describing parton cascading, and $1 + \sqrt{\alpha_{\rm s}}$ terms in the normalisation (coefficient functions). This allows one to predict the LHS of (9) up to the NNLO correction the absolute magnitude of which is of the order of

$$(\text{ LHS}) - (\text{ LHS})^{\rm MLLA} = \mathcal{O}\left(\alpha_{\rm s}(W) \cdot N_{q\bar{q}}(W)\right). \quad (10)$$

The steep growth with energy of the multiplicity factor $N(W)$ (faster than any power of $\ln W$) makes the neglected $\alpha_{\rm s}N(W)$ correction *dominate* over the (presumably) finite RHS in (9), thus endangering the very possibility of discriminating between Q- and q-jet multiplicities.

However, examining the origin of perturbative corrections proportional to $N(W)$ in (10) one can see that all of them prove to be independent of the quark mass M, being inherent to the light-quark jet evolution itself. For example, the first corrections of the order of $\alpha_{\rm s}(W)N(W)$ to the MLLA expression (10) come either from further improvement of the description of the anomalous dimension $\Delta\gamma(\alpha_{\rm s}) \sim \alpha_{\rm s}^{3/2}$ determining intrajet cascades, or from $O(\alpha_{\rm s}(W))$ terms in the coefficient function due to
(1) the three-jet configuration *quark + antiquark + hard gluon at large angle*, and
(2) the so-called "dipole correction" to the AO scheme *quark + antiquark + two soft gluons at large emission angles* [20].

Both are insensitive to the Θ_0 value with *power* accuracy $\mathcal{O}\left(\Theta_0^2\right) \ll 1$.

In fact, the statement that the LHS of (9) does not depend on the annihilation energy follows from general considerations and should hold *in all orders* in perturbation theory with power accuracy, $1 + \mathcal{O}\left(M^2/W^2\right)$.

This is a very powerful statement which goes beyond the standard renormalisation group (RG) wisdom about separation of two parametrically different scales, W and M. Indeed, by looking upon the particle multiplicity as a moment ($N = 0$) of the inclusive fragmentation function, and by drawing an analogy with the OPE analysis of DIS structure functions (space-like parton distributions), one could expect for light- and heavy-quark-initiated multiplicities

$$\frac{N_{q\bar{q}}(W)}{N_{Q\overline{Q}}(W)} = f(M) = \text{const}(W), \quad (11)$$

that is that their *ratio* rather than the *difference* is W independent. The RG motivated expectation (11) would have been correct if the quark mass M played the rôle of the initial condition for parton evolution – the *transverse momentum cut-off*. This is true enough for *hard* gluons with energies $x = 2\omega/W \sim 1$ for which the region $k_\perp < M$ is indeed suppressed as compared to the massless quark case. It is not hard gluons that dominate the accompanying multiplicity however.

Turning to (primary) gluons with $x \ll 1$ we observe that the radiation off light and heavy quarks *remains the same* down to much smaller transverse momentum scales namely,

$$k_\perp \gtrsim \omega \cdot \frac{2M}{W} = xM \ll M,$$

which is nothing but the statement of the "dead cone" suppression discussed above. It is important to stress that, being based on the analysis of the first order gluon radiation matrix element, this conclusion is *exact* and holds *in all orders* in perturbative expansion. This follows from the fact that emission of gluons with $x \ll 1$ is governed by the Low–Burnett–Kroll theorem [21] concerning the classical nature of soft accompanying radiation (following the $\mathrm{d}x \cdot (1/x - 1)$ distribution), which holds to power accuracy; see also [22].

So the QCD coherence plays a fundamental rôle in establishing this result [23]. Since the gluon bremsstrahlung off *massive* and *massless* quarks differs only at *parametrically small angles* $\Theta \lesssim \Theta_0$, the AO (QCD coherence) then ensures that the accompanying cascading effects are limited from above by a finite factor $N(W \cdot \Theta_0) \simeq N(M)$. A rigorous proof of the statement that W dependent corrections to the RHS of (9) are power suppressed as M^2/W^2 (of subleading twist nature, in the OPE nomenclature) is lacking at the moment.

By replacing the approximate MLLA multiplicities in (10) by the experimentally observable multiplicities in (9) it becomes possible to establish a phenomenological relation between the light- and heavy-quark jets with controllable accuracy.

Thus, the difference in the mean charged multiplicities, $\delta_{Q\ell}$, between heavy- and light-quark events at fixed annihilation energy W depends only on the heavy-quark mass M and remains W independent (with power accuracy) [15,23]

$$\delta_{Q\ell} = N_Q^{\mathrm{ch}}(W) - N_q^{\mathrm{ch}}(W) = \mathrm{const}(W), \qquad (12)$$

$$\delta_{bc} = N_b^{\mathrm{ch}}(W) - N_c^{\mathrm{ch}}(W) = \mathrm{const}(W), \qquad (13)$$

with $Q = b, c$ and $\ell \equiv q = u, d, s$.

Right-hand side

The RHS of (9) is estimated with MLLA accuracy. In general, this constant difference is proportional to $N(M)$ and can be given in terms of the series in $\sqrt{\alpha_s(M)}$ as the pQCD expansion parameter. Let us remark that such an expansion formally relies upon treating the quark mass M as the second hard scale, $\alpha_s(M) \ll 1$, and is bound to be only moderately satisfactory at best, since in practice, in (9), the bottom quark mass translates into $W_0^b \sim 8\,\mathrm{GeV}$ and the charm quark mass into $W_0^c \sim 2.5\,\mathrm{GeV}$ only. [3]

2.3 Quark mass effects in three-jet events

Another powerful, and phenomenologically interesting, consequence of QCD coherence is that the structure of particle cascades in three-jet $Q\overline{Q}g$ events (with a hard gluon radiated at large angle) must be identical to that in the light-quark case everywhere, apart from the two narrow angular regions corresponding to the dead cones of the Q-quarks. More specifically, the particle multiplicity in 3-jet events can be written in MLLA as the sum of quark and gluon jet multiplicities [26,27]

$$N_{q\bar{q}g}(W) = N_{q\bar{q}}(2E_q^*) + \frac{1}{2} N_{gg}(p_\perp^*), \qquad (14)$$

where E_q^* denotes the q or $\bar{q}$ energy and $p_\perp^*$ the gluon transverse momentum, both in the CMS frame of the $q\bar{q}$ pair. Then, with $W_{q\bar{q}} = 2E_q^*$, we obtain

$$N_{Q\overline{Q}g}(W) - N_{q\bar{q}g}(W) = N_{Q\overline{Q}}(W_{Q\overline{Q}}) - N_{q\bar{q}}(W_{q\bar{q}}). \qquad (15)$$

This may provide another handle for the detailed studies of the dead cone phenomenon at the reduced effective CMS energies $W_{Q\overline{Q}}$.

2.4 Discussion and estimate of next-to-MLLA terms of the order of $\alpha_s(M)N(M)$

In [28] (9) was evaluated exactly which constituted an attempt to improve the pQCD prediction beyond the $\sqrt{\alpha_s}$ accuracy beyond which (9) does not actually hold. However, the main assumption of [28] that the companion multiplicity is generated by a single cascading gluon is not valid at this level.

Next-to-MLLA correction terms are copious and it is hard to collect them all. There are, however, some specific contributions that look *enhanced* and can be listed and estimated. These are contributions containing an additional (semi-dimensional) factor π^2.

In particular, to predict the event multiplicity at the $\alpha_s N$ level, one has to take into consideration large angle two soft gluon systems (aforementioned dipole configurations). This problem is discussed in Appendix A where we show how a π^2 enhanced correction emerges. Another correction of similar nature comes from the $1 - z$ rescaling of the argument of the dead cone subtraction. This contribution is also extracted and analysed in Appendix A. It turns out to be numerically larger than the "dipole" contribution. These enhanced next-to-MLLA effects work in the same direction: they all tend to *increase* the difference between the light- and heavy-quark-initiated multiplicities in (9).

3 Theoretical predictions confronted with experiment

3.1 Experimental results on heavy-quark multiplicities in e^+e^- annihilation

The experimental measurements of hadron multiplicities in $b\bar{b}$ and $c\bar{c}$ events produced in e^+e^- annihilation were performed in the wide range of CMS energies $\sqrt{s} \equiv W$ from PEP, at $\sqrt{s} = 29\,\mathrm{GeV}$, to LEP2 at $\sqrt{s} = 206\,\mathrm{GeV}$ [29–45]. For reviews on this topic see, for instance, [46–48]. Within the experimental uncertainties the data clearly show that the differences $\delta_{b\ell}$ and $\delta_{c\ell}$ are fairly independent of the CMS energy, as expected from the perturbative analysis, and in a marked contrast to the steeply rising total mean multiplicity $N_{\mathrm{had}}^{\mathrm{ch}}$.

[3] Short-lived top quarks do not follow this pattern in the first place; the $N_{t\bar{t}}$ notion being elusive; see for example [1,24,25].

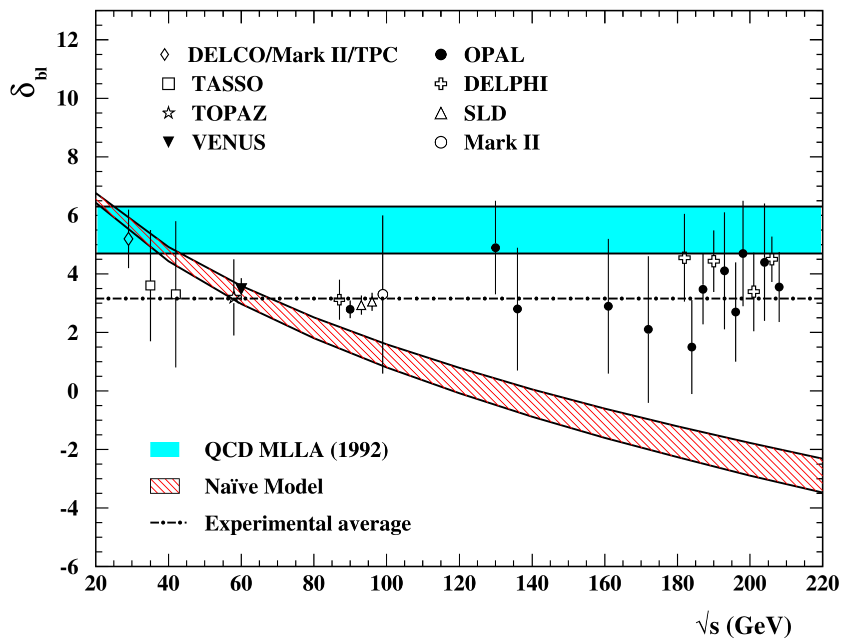

Fig. 1. Experimental measurements of $\delta_{b\ell}$ plotted as a function of the CMS energy [29–45]. The 1992 MLLA expectation $\delta_{b\ell}^{\rm MLLA} = 5.5 \pm 0.8$ [15] (shaded area) includes experimental errors on $n_b^{\rm dc}$ and light-quark multiplicities at $\sqrt{s} \simeq 8\,$GeV. The prediction of the "naïve model" [16] based on the reduction of the energy scale is also shown (dashed area)

This can be seen for example in Fig. 1, which shows a compilation of direct measurements of $\delta_{b\ell}$, (12). This figure is taken from [44] with the addition of the result from the VENUS experiment at $\sqrt{s} = 58\,$GeV [36] as well as the preliminary result from DELPHI at $\sqrt{s} = 206\,$GeV [49]. The dash-dotted line shown in Fig. 1 corresponds to the weighted average among all published results, $\delta_{b\ell}^{\rm exp} = 3.12 \pm 0.14$, assuming that the measurements are uncorrelated.

It is worthwhile to mention that the first preliminary data on the multiplicity difference between the b-quark and $udsc$-quark large angle 3-jet events produced in Z^0 decays, are reported by DELPHI [50,51]. According to (15) these results can be related to the multiplicity difference, $\delta_{bq'}$, between the b-quark and the q'-quark ($q' = u, d, s, c$) events in e^+e^- annihilation measured in the effective energy range $W_{Q\overline{Q}} \sim 53$–$59\,$GeV.

The data points do not show any sizeable energy dependence and are consistent with the precise direct result from the VENUS experiment [36] at $\sqrt{s} = 58\,$GeV.

As it can be seen in Fig. 1, within the experimental uncertainties most data points are consistent with the original MLLA prediction [15], $\delta_{b\ell}^{\rm MLLA} = 5.5 \pm 0.8$. However, the precise results from the OPAL, SLD, DELPHI and VENUS experiments, which dominate the weighted average value $\delta_{b\ell}^{\rm exp}$, are definitely lower. The "naïve model", based on the reduction of the energy scale $\sqrt{s}$, which predicts the growing difference as in (3) and, therefore, the gradually decreasing $\delta_{b\ell}$ is strongly disfavoured.

3.2 Test of MLLA predictions for b-quark jets

Our main goal here is to explain why the previous numerical value of the MLLA prediction of $\delta_{b\ell}^{\rm MLLA} = 5.5 \pm 0.8$ [15] needs a revision. This value relies strongly on experimentally measured quantities, and some new relevant results became available since the analysis presented in [15]. Furthermore, we reanalysed the old data on charged multiplicities at low energies, which in addition to some small errors propagated in the literature until now affected the result presented in [15].

As we already mentioned, the difference between the MLLA result and the experimental data on $\delta_{b\ell}$ would allow one to probe the size of the next-to-MLLA effects of order $\alpha_{\rm s}(M_b)N_{q\bar{q}}(M_b)$.

Let us first take a fresh look at the MLLA expression for the charged multiplicity difference $\delta_{b\ell}$,

$$\delta_{b\ell}^{\rm MLLA} = n_b^{\rm dc} - N_{q\bar{q}}^{\rm ch}(\sqrt{e}M_b), \qquad (16)$$

in order to establish whether and where the two terms in the RHS of (16) require revision in the light of the current improvements in the understanding of experimental data.

First we consider the mean heavy hadron charged decay multiplicity, $n_b^{\rm dc}$. In the analysis of [15] the average number of charged particles coming from the decay of two B-hadrons was taken as $n_b^{\rm dc} = 2N_b^{\rm dc} = 11.0 \pm 0.2$. In the present analysis we used the most recent result obtained from the combination of the ALEPH, CDF, DELPHI, L3, OPAL and SLD data on B-hadron production [52], $N_b^{\rm dc} = 4.955 \pm 0.062$, with an addition of 0.485 ± 0.065 tracks to include the charged decay products of K_s^0 and Λ, as measured by OPAL [53]. There is still an issue of the role of heavier B-hadron states ($B^*, B^{**}, \ldots$) and on how fast the "saturation" with rising energy can be established. Their contribution to the mean heavy hadron charged decay multiplicity, $n_b^{\rm dc}$ is usually evaluated with the help of Monte Carlo models. We used the value 0.22 quoted by the SLD experiment [42], which should be almost independent on $\sqrt{s}$ for CMS energies above the Z^0 mass peak.

We finally arrive at the value

$$n_b^{\rm dc} = 11.10 \pm 0.18, \qquad (17)$$

which practically coincides with the previous result in [15].

Second, on the subtraction term $N^{\rm ch}_{q\bar{q}}(\sqrt{e}M_b)$ we note the following. The second term in (16) is related to the radiation within the dead cone, where primary gluons emitted off the b-quark and the b-quark itself act as a source of secondary soft radiation. In order to quantify the size of this term we have to address first the issue of the definition of the b-quark mass, which should be appropriate for the dead cone physics.

As is well known, within perturbative calculations it would be natural to take the pole in the quark propagator as the definition of the quark mass. By its very construction the pole mass is directly related to the concept of the free quark mass. However, due to the infrared effects the pole mass cannot be used with arbitrary high accuracy (see [54] for recent review and references). Though in a more sophisticated calculation a mass definition which is less sensitive to the small momenta may appear to be more appropriate, the uncertainties in the quark mass of order of $\Lambda_{\rm QCD}$ are far beyond the accuracy of our consideration here. So for the purposes of this paper we use the two-loop pole mass value, quoted in [5],

$$(M_b)_{\rm pole} = 4.7\ –5.0\,{\rm GeV}, \tag{18}$$

which cover, in particular, some of the short distance mass prescriptions [55, 56]. The scale $W^b_0 = \sqrt{e}M_b$ at which the subtraction term $N^{\rm ch}_{q\bar{q}}(W^b_0)$ must be evaluated is then $\sqrt{s} = (8.0 \pm 0.25)$ GeV.

Since there are no direct measurements of charged multiplicity at this energy, we estimate $N^{\rm ch}_{q\bar{q}}(8\,{\rm GeV})$ in the following way.
(1) Use as many as possible experimental results on inclusive mean charged multiplicity $N^{\rm ch}_{\rm had}$ below and above $\sqrt{s} = 8.0$ GeV, rather than restricting to a very limited energy range as in [15];
(2) fit the data points to evaluate $N^{\rm ch}_{\rm had}(8\,{\rm GeV})$ by interpolation, using different parameterisations and over a wide energy range, in order to test the consistency and stability of the results and to estimate a reasonably conservative uncertainty for $N^{\rm ch}_{\rm had}(8\,{\rm GeV})$;
(3) evaluate $N^{\rm ch}_{q\bar{q}}(8\,{\rm GeV})$ by subtracting the c-quark contamination from $N^{\rm ch}_{\rm had}(8\,{\rm GeV})$.

We studied all available data on the mean charged particle multiplicity, $N^{\rm ch}_{\rm had}$, collected in e^+e^- annihilations in the centre-of-mass energy range 1.4–91 GeV. We considered only published results obtained in the continuum, thus away from the J/Ψ and Υ resonances, which were determined following what is now considered a standard convention [57], namely including in the evaluation of the mean value all charged particles produced in the decays of particles with lifetimes shorter than $3\cdot 10^{-10}$ s [58–74]. This means that the charged decay products of K^0_s and of weakly decaying heavy-mesons $(D, B, \ldots)$ and baryons $(\Lambda, \Sigma, \ldots)$ as well as of their antiparticles must be considered, regardless of how far away from the interaction point the decay actually occurs. Unfortunately, some old publications, particularly those obtained at energies below 7 GeV, do not explain sufficiently well how the data were treated in this respect. We use only the ones which clearly considered at least charged decay products of K^0_s, that is known to be the dominant contribution at low energies. Furthermore, we do not consider results obtained at energies which might suffer from threshold effects due to charmed meson pair production, including higher mass states, notably the data collected by MARK I [59] in the interval 4.0 to 7.0 GeV. In order to evaluate $N^{\rm ch}_{\rm had}(8.0\,{\rm GeV})$ we fit the data points using the following parameterisations:

$$N^{\rm ch}_{\rm had} = a + b\cdot\ln(s) + c\cdot\ln^2(s), \tag{19}$$

$$N^{\rm ch}_{\rm had} = a\cdot s^b, \tag{20}$$

$$N^{\rm ch}_{\rm had} = a\cdot\alpha_{\rm s}^{\beta}\cdot\exp(\gamma/\sqrt{\alpha_{\rm s}}), \tag{21}$$

which are known to describe the data on the mean charged multiplicity very well [75]. The parameters a, b and c, as well as the effective scale Λ, not explicitly shown in (21) but entering the definition of the running coupling $\alpha_{\rm s}$, are free parameters.[4] The two cases with three and five active flavours were considered in the calculation of $\alpha_{\rm s}$ when making fits.

We tested the consistency and the stability of the results by varying the fit energy range over the intervals: 7–14 GeV; 7–44 GeV (to include the results from PEP and PETRA); 7–62 GeV (to include results from TRISTAN) and 7–91.2 GeV (to include results from LEP1 and SLC), the common starting point of 7 GeV being well above the charmed meson production threshold. All mean multiplicities measured above $\sqrt{s} = 10.5$ GeV were corrected for the effects caused by the b-quark. At each energy the correction was applied accounting for the fractions of the various quark species as predicted by the standard model and using the value $\delta_{b\ell} = 3.1$ as measured experimentally. The total uncertainty associated with each data point was taken as the statistical and the systematic uncertainties added in quadrature.

All fits give a very good χ^2, and the mean charged multiplicity predicted at $\sqrt{s} = 8$ GeV is found to vary between 6.9 and 7.3.

Our conclusion is that in the energy interval $\sqrt{s} = 7.75$–8.25 GeV

$$N^{\rm ch}_{\rm had}(8.0\,{\rm GeV}) = 7.1 \pm 0.3. \tag{22}$$

The uncertainty includes the observed spread of values due to the choice of different parameterisations as well as the effect due to the uncertainty of the b-quark pole mass, $(M_b)_{\rm pole}$.

The result presented in (22), however, refers to a mixture of u, d, s, c events, while for the determination of $\delta^{\rm MLLA}_{b\ell}$ from (16) and (17) only the contribution to $N^{\rm ch}_{\rm had}$ from the light quarks ($q = u, d, s$), $N^{\rm ch}_{q\bar{q}}$, should be considered. In order to extract the light-quark event multiplicity from (22) we carefully studied the literature about the experimental results on the measurement of the multiplicity difference between the q- and c-quarks

$$\delta^{\rm exp}_{c\ell} = N^{\rm ch}_{c} - N^{\rm ch}_{q\bar{q}}. \tag{23}$$

[4] Note that only (21) is pQCD motivated, but we are using (19) and (20) as well for interpolation purposes and error evaluation.

At the time of the analysis of [15] only the results from MARK II, TPC and TASSO were available. These results are affected by large uncertainties, and we also noticed in the literature some inconsistencies in the evaluation of $\delta_{c\ell}^{\rm exp}$, which we corrected for. Much more precise results from OPAL [41] and SLD [42,45] are now available, and in the present analysis the experimental value of $\delta_{c\ell}^{\rm exp}$ to be used for the correction was reevaluated, as discussed in detail in Appendix B.1. It is shown there that the experimental results from 29 GeV to 91 GeV are well consistent with a constant value, and a weighted average yields

$$\delta_{c\ell}^{\rm exp} = 1.0 \pm 0.4. \qquad (24)$$

This value is about a factor two smaller than that used in [15], $\delta_{c\ell} = 2.2 \pm 1.2$, and is more precise. Since no direct measurements of $\delta_{c\ell}$ at $\sqrt{s} = 8$ GeV exist, we assume its constancy also at lower energies, as in [15]. Clearly, a direct and accurate measurement of $\delta_{c\ell}(\sqrt{s} = 8)$ GeV, for example by analysing radiative events with the proper effective energy at the BaBar and Belle experiments, would be very desirable to validate our hypothesis.

We finally correct $N_{\rm had}^{\rm ch}$ for the effect of the 40% admixture of $c\bar{c}$ events using this new result on $\delta_{c\ell}^{\rm exp}$, and find for the light quarks

$$N_{q\bar{q}}^{\rm ch}(8.0\,{\rm GeV}) = 6.7 \pm 0.34. \qquad (25)$$

As a cross-check of this method, we estimate $N_{q\bar{q}}^{\rm ch}$ also in the following way. Besides the b-quark contribution, we subtract also the c-quark contribution from all the mean charged multiplicities measured above the c-quark threshold. This is done using the value of $\delta_{c\ell}^{\rm exp}$ presented in (24) and the standard model predictions for the c-quark fractions at each energy.

With the exception of the MLLA parameterisation which in principle should not be used below the b-quark threshold, we can then extend the fitting procedure described above to the published results down to 1.4 GeV. This time the interpolation at 8 GeV provides directly the evaluation of $N_{q\bar{q}}^{\rm ch}$, to be used for the calculation of $\delta_{b\ell}^{\rm MLLA}$. The values of $N_{q\bar{q}}^{\rm ch}(8\,{\rm GeV})$ are found to range in the interval 6.45–6.65, completely consistent with the value of 6.7 ± 0.34 quoted in (25).

We also compared our findings with the results from several global QCD fits to $N_{\rm had}^{\rm ch}$ at 8.0 GeV. The numerical solution of the MLLA evolution equation for the particle multiplicity generated by light quarks, supplemented by the full $O(\alpha_{\rm s})$ effects for e^+e^- annihilation [76], gives $N_{\rm had}^{\rm ch}(8.0\,{\rm GeV}) = 6.5$. In this fit no effort was undertaken to separate the contributions from different flavours, so the fit which includes low energy data as well should be placed in between $N_{\rm had}^{\rm ch}$ and $N_{q\bar{q}}^{\rm ch}$, to be compared with (22) and (25). The 3NLO fit [77] using the data above 10 GeV gives $N_{\rm had}^{\rm ch}(8.0\,{\rm GeV}) = 7.3$, consistent with (22). Furthermore, a value $N_{q\bar{q}}^{\rm ch}(8.0\,{\rm GeV}) = 6.5$ is found by running the Pythia 6.2 Monte Carlo program (in its default version) with light quarks only, with initial state radiation switched off and following the standard convention for the definition of mean charged multiplicity [79], in good agreement with our result (25).

Substituting (17) and (25) into (16) we arrive at the revised MLLA expectation for the multiplicity difference

$$\delta_{b\ell}^{\rm MLLA} = 4.4 \pm 0.4, \qquad (26)$$

which is ~ 1.0 unit lower than the result reported in [15] and has half of its uncertainty.

The comparison of the MLLA result (26) with the available experimental data on $\delta_{b\ell}$ in e^+e^- annihilation is shown in Fig. 2; here we included also the reevaluated results of DELCO, MARK II, TPC, TASSO, TOPAZ and VENUS (see Appendix B.2). The new experimental average is given by

$$\delta_{b\ell}^{\rm exp} = 3.14 \pm 0.14. \qquad (27)$$

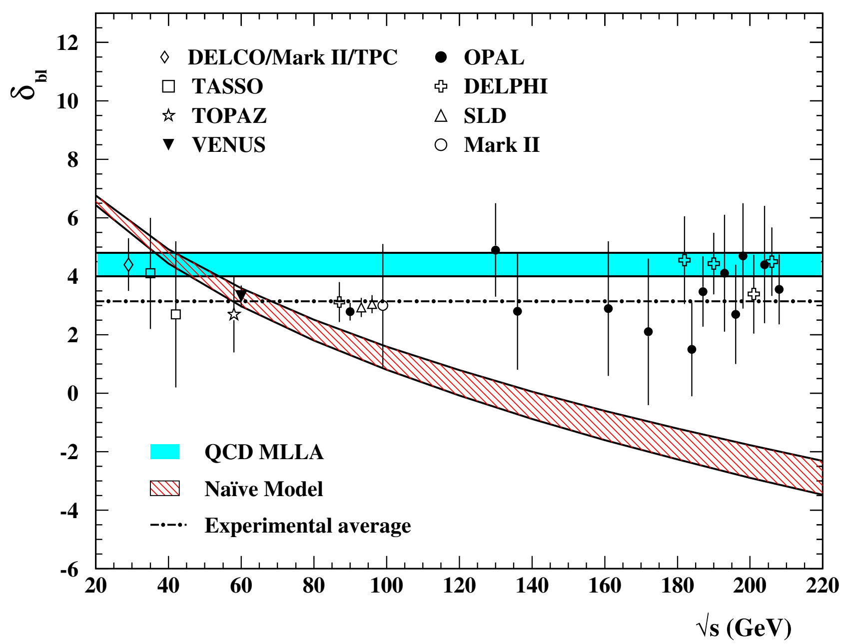

Fig. 2. Experimental measurements of $\delta_{b\ell}$ plotted as a function of the CMS energy, $\sqrt{s}$; data below 90 GeV reevaluated (see Appendix B.2). The revised MLLA expectation using $\delta_{b\ell}^{\rm MLLA} = 4.4 \pm 0.4$ is indicated by the shaded area. Also shown is the "naïve model" [16] based on the reduction of energy scale (dashed area)

We can say that, qualitatively, the previous conclusion that the experimental mean value is lower than the absolute value of the MLLA prediction remains valid. Quantitatively, however, the agreement between the data and the theory definitely improves.

Finally, we turn to the question of whether the remaining discrepancy can be attributed to the next-to-MLLA contributions. First we note that the experimental value of the multiplicity difference $N_{q\bar{q}}(W) - N_{b\bar{b}}(W) = n_b^{\rm dc} - \delta_{b\ell}^{\rm exp} = 7.96 \pm 0.23$ and the MLLA expectation $N_{q\bar{q}}(\sqrt{e}M)$ as evaluated in (25) differ by a relative amount $< 20\%$ which is of the order of the expected correction term in (9) of $\mathcal{O}\left(\alpha_{\rm s}(M)\right)$. To gain insight at the quantitative level, we consider first the size of the above multiplicity difference in the double logarithmic approximation (DLA). This is given by (A.30) but with the RHS replaced simply by $N_{q\bar{q}}(M)$ if the dominant contribution to N_0 in (A.11) is taken. For b-quark jets, this requires evaluation of the multiplicity at $M_b \sim 4.85\,\text{GeV}$. We estimate $N_{\rm had}^{\rm ch}(M_b) \approx 5.1$ ($N_{q\bar{q}}^{\rm ch}(M_b) \approx 4.7$) and, therefore, $\delta_{b\ell}^{\rm DLA} \sim 6.4$. This is about two units above the MLLA prediction (26) which, in turn, is about one unit above the data in (27) indicating convergence.

In the next-to-MLLA two large "π^2-contributions" are derived explicitly; see (A.30) in Appendix A. The final expression involves the coupling at scale M which we derive from the 1-loop formula with $\Lambda = 250\,\text{MeV}$, as typically used in MLLA applications (see for example [6,20]), and we obtain $\alpha_{\rm s}(M_b) = 0.23$ for $n_f = 3$ flavours. Then from (A.30) we find $N_{q\bar{q}}^{\rm ch}(W) - N_{Q\bar{Q}}^{\rm ch}(W) = N_{q\bar{q}}^{\rm ch}(\sqrt{e}M) \times 1.27 \approx 8.5$. This finally gives the result including these next-to-MLLA contributions $\delta_{b\ell} \approx 2.6 \pm 0.4$. We, therefore, conclude that the MLLA prediction is already close to the experimental data in (27), and the remaining difference is of the order of the expected next-to-MLLA contributions.

3.3 Results on charm quark jets

Since the scale relevant for the charm quark, $W_0^c \sim \sqrt{e}M_c$, is significantly lower than in the b-quark case the predictions are less reliable.

The two-loop c-quark pole mass is quoted in [5] as

$$(M_c)_{\rm pole} = 1.47 \text{ –}1.83 \text{ GeV}. \qquad (28)$$

We evaluated the size of the subtraction term $N^{\rm ch}(W_0^c)$ where we followed the same strategy as described above, restricting the multiplicity fits to the energy range 1.4–10.45 GeV. The predicted value at $W_0^c = 2.7\,\text{GeV}$ is found to vary between 3.5 and 3.9, and we arrive at $N_{q\bar{q}}^{\rm ch}(2.7\,\text{GeV}) = 3.7 \pm 0.3$. Using the c-quark decay multiplicity $n_c^{\rm dc} = 5.2 \pm 0.3$ we obtain the MLLA expectation for the charged particle multiplicity difference in the c-quark case,

$$\delta_{c\ell}^{\rm MLLA} = 1.5 \pm 0.4, \qquad (29)$$

which is basically the same as the previous number $\delta_{c\ell}^{\rm MLLA} = 1.7 \pm 0.5$ in [15]. The result (29) is consistent with the new more precise experimental average given by (24). As in the case of $\delta_{b\ell}$ the theoretical MLLA result lies now above the experimental value which is expected due to the presence of the higher order effects.

We also note an interesting aspect of the difference between the b- and c-quark multiplicities $\delta_{bc} = \delta_{b\ell} - \delta_{c\ell}$. Since the M dependence of the next-to-MLLA term in (A.30) is weaker than that of the leading $N_{q\bar{q}}(\sqrt{e}M)$ contribution, the multiplicity difference $\delta_{cb} = \delta_{b\ell} - \delta_{c\ell}$ is less affected by this correction and can be better approximated by the MLLA result. If we compare the experimental and theoretical numbers obtained from the results derived above,

$$\delta_{bc}^{\rm MLLA} = 2.9 \pm 0.6, \qquad \delta_{bc}^{\rm exp} = 2.1 \pm 0.4, \qquad (30)$$

we find indeed that, contrary to the difference $\delta_{b\ell}$, within the slightly larger errors, there is a reasonable agreement between the data and the MLLA prediction for this multiplicity difference involving b-quarks.

4 Conclusions

The comparison of particle multiplicities in heavy- and light-quark-initiated jets provides a specific test of the perturbative approach to multiparticle production. In this approach the particle multiplicities in e^+e^- annihilation are directly proportional to the gluon multiplicities generated by multiple successive bremsstrahlung processes from the primary quarks. In the case of a primary heavy quark the small angle radiation is kinematically suppressed (dead cone effect). Also the subsequent gluon emission is affected by the mass effects, which results in the loss of coherence of soft gluon radiation off the heavy quark and the primary gluon. The result can be represented as an appropriate expansion in $\sqrt{\alpha_{\rm s}}$ where the leading double logarithmic and next-to-leading (MLLA) terms have been known for quite a while, whereas certain large contributions in the next-to-MLLA order are discussed here.

The main aim of this study is to sharpen the tests of the perturbative approach by accounting for all currently available data on e^+e^- annihilation. More accurate theoretical predictions for the difference of multiplicities in light- and heavy-quark jets are obtained. The expected energy independence of this difference is nicely confirmed. The same difference in 3-jet events is expected to agree with that in 2-jet events at the corresponding $q\bar{q}$ CMS energy, and this is supported by preliminary data. As compared to the previous analysis, the updated MLLA prediction for the absolute value of the multiplicity difference comes closer to the experimental data. It is shown that the remaining difference is of the order of the next-to-MLLA corrections considered. This way the specific effects related to soft gluon bremsstrahlung from heavy quarks and their impact on the generation of the gluon cascade are reaching quantitative understanding within the perturbative approach.

It would be very interesting to extend the measurements of $\delta_{b\ell}$ and $\delta_{c\ell}$ to lower energies, for example down to the region accessible at the B-factories. In particular, a direct measurement of $\delta_{c\ell}$ at $\sqrt{s} = 8.0\,\text{GeV}$, for example from the analysis of events with initial state photon radiation, would be very important to confirm our assumption that $\delta_{c\ell}$ remains constant below 29 GeV. Further tests of the QCD predictions at higher energies at a future linear collider will be interesting.

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Appendix A

A.1 Single gluon emission in MLLA and beyond

The exact first order expression for the probability of single gluon emission off the heavy-quark pair can be written in the following form [3, 22], in analogy with QED [78],

$$\mathrm{d}w_V = \frac{C_F\alpha_s}{\pi v}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}\eta}{\sqrt{1-\eta}} \times \left\{2(1-z)\frac{\eta-\eta_0}{\eta^2} + z^2\left[\frac{1}{\eta}-\frac{1}{2}\right]\zeta_V^{-1}\right\}, \tag{A.1}$$

with z the gluon energy fraction and η an angular variable and

$$1 \ge \eta = 1-\beta^2\cos^2\Theta_c \ge \eta_0 = \frac{4m^2}{1-z}, \quad m \equiv \frac{M}{W} \ll 1, \tag{A.2}$$

where β is the quark velocity and Θ_c is the polar gluon angle in the $Q\overline{Q}$ CMS,

$$\beta^2 = \beta^2(z) = 1 - \frac{4m^2}{1-z} \le v^2 = 1-4m^2 \ge z. \tag{A.3}$$

The first term in curly brackets in (A.1) contains the main (double logarithmic) contribution and corresponds to universal soft gluon bremsstrahlung. In accordance with the Low–Barnett–Kroll theorem [21], both the $\mathrm{d}z/z$ and $\mathrm{d}z$ parts of the radiation density have a classical origin and are, therefore, universal, independent of the process (and of the quark spin). This term explicitly exhibits the dead cone phenomenon: "soft" radiation vanishes in the forward direction, $\sin\Theta_c \to 0$, $\eta \to \eta_0$.

The second term proportional to $\mathrm{d}zz$ (hard gluons) depends, generally speaking, on the $Q\overline{Q}$ production mechanism. Namely, both the $-1/2$ subtraction term and the factor $\zeta_V = (3-v^2)/2 = 1+2m^2$ would be different for production current other than the vector current. We include this remark to stress that at the level of α_s corrections (as well as of power suppressed effects $\mathcal{O}(\sqrt{\alpha_s}m^2)$) the mean multiplicity acquires process dependent contributions from 3-jet ensembles and cannot be treated any longer as an intrinsic characteristic of the $Q\overline{Q}$ system.

To obtain the mean parton multiplicity with the MLLA accuracy it suffices to supply (A.1) with the gluon cascading factor which depends, together with the running coupling, on the argument [3]

$$k_t^2 = \left(\frac{zW}{2}\right)^2\eta. \tag{A.4}$$

Neglecting relative corrections $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(m^2)$ we obtain for the mean multiplicity

$$N_{Q\overline{Q}}(W) = N_0 - N_1, \tag{A.5}$$

where

$$N_0 = \frac{C_F}{\pi}\int_0^{v^2}\frac{\mathrm{d}z}{z}\int_{\eta_0}^1\frac{\mathrm{d}\eta}{\eta}\frac{1+(1-z)^2}{\sqrt{1-\eta}}\cdot[\alpha_s\mathcal{N}_G](k_t), \tag{A.6}$$

$$N_1 = \frac{C_F}{\pi}\int_0^{v^2}\frac{\mathrm{d}z}{z}\int_{\eta_0}^1\frac{\eta_0\mathrm{d}\eta}{\eta^2}2(1-z)\cdot[\alpha_s\mathcal{N}_G](k_t). \tag{A.7}$$

We start by analysing the leading term (A.6) for N_0. The kinematical factor $1/\sqrt{1-\eta}$ somewhat enhances the contribution of the large angle region, $\eta = \mathcal{O}(1)$, and should be taken into consideration in the leading DL term. The corresponding (SL) correction can be approximately accounted for by pushing up the upper limit of the logarithmic integration. Indeed, given that the factor $\mathcal{F} \equiv \alpha_s\mathcal{N}$ depends on η logarithmically, the chain of approximations follows reading

$$\begin{aligned}&\int_{\eta_0}^1\frac{\mathrm{d}\eta}{\eta\sqrt{1-\eta}}\mathcal{F}(\eta)\\ &= \int_{\eta_0}^1\frac{\mathrm{d}\eta}{\eta}\mathcal{F}(\eta) + \int_{\eta_0}^1\frac{\mathrm{d}\eta}{\eta}\left(\frac{1}{\sqrt{1-\eta}}-1\right)\mathcal{F}(\eta)\\ &\approx \int_{\eta_0}^1\frac{\mathrm{d}\eta}{\eta}\mathcal{F}(\eta) + \ln 4\cdot\mathcal{F}(1) \approx \int_{\eta_0}^4\frac{\mathrm{d}\eta}{\eta}\mathcal{F}(\eta).\end{aligned} \tag{A.8}$$

It is straightforward to check that the omitted terms are limited from above by the $\mathcal{O}(\alpha_s(W))$ and $\mathcal{O}(m^2)$ terms. Natural rescaling of the integration variable, $t = \eta W^2/4$, leads to

$$N_0 = \frac{C_F}{\pi}\int_0^{v^2}\mathrm{d}z\frac{1+(1-z)^2}{z}\int_{M^2/(1-z)}^{W^2}\frac{\mathrm{d}t}{t}[\alpha_s\mathcal{N}_G](k_t), \tag{A.9}$$

with

$$k_t^2 = z^2 t. \tag{A.10}$$

Now we represent (A.9) as

$$N_0 = N_{q\bar{q}}(W) - N_{q\bar{q}}(M) - N_2, \tag{A.11}$$

where we have singled out an (enhanced) next-to-MLLA correction term that we will consider later,

$$\begin{aligned}N_2 &= N_2(M) \\ &= \frac{C_F}{\pi}\int_0^{v^2}\mathrm{d}z\frac{1+(1-z)^2}{z}\int_{M^2}^{M^2/(1-z)}\frac{\mathrm{d}t}{t}[\alpha_s\mathcal{N}_G](k_t),\end{aligned} \tag{A.12}$$

and introduced the function

$$N_{q\bar q}(W) = \frac{C_F}{\pi}\int_0^1 \mathrm{d}z\,\frac{1+(1-z)^2}{z}\int^{W^2}\frac{\mathrm{d}t}{t}\left[\alpha_s\mathcal{N}_G\right](k_t)\,, \tag{A.13}$$

that describes the *light-quark* event multiplicity at the CMS energy W.

We observe that the dead cone suppression naturally results in the expression for the accompanying multiplicity in $Q\overline{Q}$ events as a difference of light-quark multiplicities at CMS energies W and M. The MLLA correction N_1 defined in (A.7) modifies the effective energy of the subtraction term $N(M)$ in (A.11).

Now for N_1. Since the η integral in (A.7) is non-logarithmic and is concentrated in the region $\eta\sim\eta_0\ll 1$, we allowed ourselves to drop the $1/\sqrt{1-\eta}$ factor here as producing a negligible $\mathcal{O}\left(m^2\right)$ correction. We have

$$N_1 = \frac{C_F}{\pi}\int_0^1 \mathrm{d}z\,\frac{2(1-z)}{z}\int_1^\infty\frac{\mathrm{d}u}{u^2}\left[\alpha_s\mathcal{N}_G\right](k_{t0}),$$

$$k_{t0}^2 = \frac{z^2}{1-z}M^2u. \tag{A.14}$$

Though the *collinear* logarithmic enhancement disappears here, the *soft* one is still present (contrary to N_2) and promotes N_1 to the $\sqrt{\alpha_s}$ (MLLA) level.

First we observe that the $(1-z)$ rescaling of the argument of the cascading factor $[\alpha_s\mathcal{N}]$ is negligible as it produces a $\mathcal{O}\left(\alpha_s^{3/2}\right)$ correction (next-to-next-to-MLLA). Then, making use of the expansion

$$\int_1^\infty\frac{\mathrm{d}u}{u^2}F(\ln u) = F(0)+F'(0)+\dots$$

and replacing the factor $2(1-z)$ by the numerator of the full quark $\to$ gluon splitting function, $1+(1-z)^2$, we arrive at

$$N_1 = \frac{C_F}{\pi}\left(\int_0^1\mathrm{d}z\,\frac{1+(1-z)^2}{z}[\alpha_s\mathcal{N}_G](z\sqrt{e}M) - \frac{1}{2}[\alpha_s\mathcal{N}_G](M)\right), \tag{A.15}$$

which holds with the next-to-MLLA accuracy (including $\mathcal{O}(\alpha_s)$). By comparing (A.15) with (A.13) and recalling (A.10) we can express the MLLA correction as the *logarithmic derivative* of the light-quark multiplicity:

$$N_1 = \frac{1}{2}N'_{q\bar q}(\sqrt{e}M)\,;\qquad N'_{q\bar q}(Q)\equiv\frac{\mathrm{d}}{\mathrm{d}\ln Q}N_{q\bar q}(Q). \tag{A.16}$$

Invoking (A.5) and (A.11), for the $Q\overline{Q}$ event multiplicity we finally obtain

$$N_{Q\overline{Q}}(W)$$
$$= N_{q\bar q}(W) - \left[N_{q\bar q}(M)+\frac{1}{2}N'_{q\bar q}(\sqrt{e}M)+\dots\right] - N_2$$
$$\simeq N_{q\bar q}(W) - N_{q\bar q}(\sqrt{e}M) - N_2. \tag{A.17}$$

This proves the MLLA subtraction formula (9).

Let us now consider N_2. The effect due to the $(1-z)$ rescaling of the lower limit of the t-integration in (A.9) produces a π^2 enhanced next-to-MLLA correction N_2. We have

$$N_2(M) \tag{A.18}$$
$$= \frac{C_F}{\pi}\int_0^{v^2}\mathrm{d}z\,\frac{1+(1-z)^2}{z}\int_{M^2}^{M^2/(1-z)}\frac{\mathrm{d}t}{t}\left[\alpha_s\mathcal{N}_G\right](k_t)$$
$$\simeq \frac{C_F}{\pi}\int_0^1\mathrm{d}z\,\frac{1+(1-z)^2}{z}\ln\frac{1}{1-z}\left[\alpha_s\mathcal{N}_G\right](M)$$
$$= \frac{C_F}{\pi}\left(\frac{\pi^2}{3}-\frac{5}{4}\right)\left[\alpha_s\mathcal{N}_G\right](M)\cdot\left\{1+\mathcal{O}\left(\alpha_s^{1/2}(M)\right)\right\}.$$

In terms of the event multiplicity ($N_{q\bar q}\simeq 2C_F/N_c\cdot\mathcal{N}_G$) we arrive at the relative correction

$$\frac{N_2(M)}{N_{q\bar q}(M)}\simeq\frac{N_c\alpha_s}{2\pi}\cdot\left(\frac{\pi^2}{3}-\frac{5}{4}\right). \tag{A.19}$$

A.2 Two gluon (dipole) correction

To derive the probabilistic MLLA equations describing parton cascades one has to analyse, in particular, ensembles of many energy ordered gluons radiated at arbitrary angles and demonstrate that, after having taken into full account multiple interference diagrams, one arrived at the pattern of *angular ordered* (AO) successive gluon emission [20]. Reduction of interference graphs to the probabilistic AO scheme is not exact: there is a "remainder". In particular, the first such remainder appears at the α_s^2 order and describes radiation of two soft gluons (with energies $k_2\ll k_1\ll W$) at large angles with respect to the $q\bar q$ pair and to each other. The angular structure of the remainder $R^{(2)}$ is as follows:

$$R^{(2)} = C_F H_+^1\cdot N_c D^2_{-[+1]} + C_F H_-^1\cdot N_c D^2_{+[-1]}, \tag{A.20}$$

where $\pm$ mark the momenta of q and $\bar q$, the factor H describes independent gluon emission,

$$H_\ell^i = \frac{2}{a_{i\ell}}, \tag{A.21}$$

$$a_{ik} = q^2\frac{(p_i\cdot p_k)}{(p_i\cdot q)(p_k\cdot q)} = 1-\mathbf{n}_i\mathbf{n}_k = 1-\cos\Theta_{ik},$$

and D is the so-called "dipole factor",

$$D^i_{\ell\,[mn]}\equiv I^i_{\ell m}-I^i_{\ell n}, \tag{A.22}$$

$$I^i_{\ell m} = \frac{a_{i\ell}+a_{im}-a_{\ell m}}{a_{i\ell}a_{im}}. \tag{A.23}$$

The dipole remainder possesses no collinear singularities,

$$\int\frac{\mathrm{d}\Omega_2}{4\pi}\int\frac{\mathrm{d}\Omega_1}{4\pi}H_+^1 D^2_{-[+1]} = \int_0^1\frac{\mathrm{d}x}{1-x}\ln x = -\zeta(2) = -\frac{\pi^2}{6}, \tag{A.24}$$

so that the integration of (A.20) over the gluon angles gives

$$\int \frac{d\Omega_2}{4\pi} \int \frac{d\Omega_1}{4\pi} R^{(2)}(\mathbf{n}_2, \mathbf{n}_1) = 2C_F N_c \cdot \left(-\frac{\pi^2}{6}\right). \quad (A.25)$$

With account of the gluon cascading factor, logarithmic integrals over the gluon energies induce the next-to-MLLA correction to the event multiplicity

$$\Delta N_{q\bar{q}}(Q) \quad (A.26)$$
$$= 2C_F N_c \left(-\frac{\pi^2}{6}\right) \int^Q \frac{dk_1}{k_1} \frac{\alpha_s}{2\pi} \int^{k_1} \frac{dk_2}{k_2} \frac{\alpha_s}{2\pi} \mathcal{N}_G(k_2).$$

Now we estimate the energy integrals using

$$\int^k \frac{dk'}{k'} \mathcal{N}_G(k') \simeq \frac{1}{\gamma_0} \cdot \mathcal{N}_G(k), \qquad \gamma_0 = \sqrt{\frac{2N_c\alpha_s}{\pi}},$$

with γ_0 the DLA multiplicity anomalous dimension, and obtain another π^2 enhanced relative correction:

$$\frac{\Delta N_{q\bar{q}}(Q)}{N_{q\bar{q}}(Q)} = -N_c^2 \frac{\pi^2}{6} \frac{(\alpha_s/2\pi)^2}{\gamma_0^2} = -\frac{N_c\alpha_s(Q)}{2\pi} \cdot \frac{\pi^2}{24}. \quad (A.27)$$

This means that the true multiplicity $N_{q\bar{q}} = N_{q\bar{q}}^{(\text{ MLLA})} + \Delta N_{q\bar{q}}$ and its MLLA estimate are related as follows:

$$N_{q\bar{q}}^{(\text{MLLA})}(Q) \simeq N_{q\bar{q}}(Q) \cdot \left(1 + \frac{N_c\alpha_s(Q)}{2\pi} \cdot \frac{\pi^2}{24}\right). \quad (A.28)$$

Now we return to the expression (A.17):

$$N_{q\bar{q}}(W) - N_{Q\overline{Q}}(W) = N_{q\bar{q}}(\sqrt{e}M) + N_2(M). \quad (A.29)$$

The first observation we make is that in the difference $N_{q\bar{q}}(W) - N_{Q\overline{Q}}(W)$ the two-gluon dipole corrections cancel since, as we discussed above, large angle soft gluon emission is insensitive to quark mass. Therefore, we can look upon the LHS as being constructed of the true multiplicities.

On the contrary, the factor $N_{q\bar{q}}$ on the RHS of (A.29) is the theoretical (MLLA) expression. Relating it with the true multiplicity via (A.28) results in

$$N_{q\bar{q}}(W) - N_{Q\overline{Q}}(W) \quad (A.30)$$
$$= N_{q\bar{q}}(\sqrt{e}M) \cdot \left\{1 + \frac{N_c\alpha_s(M)}{2\pi}\left[\frac{\pi^2}{24} + \left(\frac{\pi^2}{3} - \frac{5}{4}\right)\right]\right\},$$

where we inserted the expression (A.19) for the first enhanced correction N_2. Numerically, the first term in the square bracket from the dipole corrections at large emission angles amounts only to about 4% whereas the second one, which improves the description of the small angle emission from the heavy quark, is about 5 times larger. The result (A.30) is not claimed to be complete at this order but it includes the important π^2 contributions considered to be dominant and shows the size of the next-to-MLLA terms. Remarkably, both corrections work in the same direction increasing the difference between the light- and heavy-quark companion multiplicities.

Appendix B

B.1 On the measurement of $\delta_{c\ell}$

The experimental determination of $\delta_{c\ell}$ at different energies is very important for this analysis. As discussed in Sect. 3.2, a key point in the evaluation of the absolute value of the MLLA prediction for $\delta_{b\ell}$ is the determination of the light-quark mean multiplicity, $N_{q\bar{q}}^{\text{ch}}$, at $\sqrt{s} = 8$ GeV. Experimentally one measures the mean charged multiplicity of an unbiased inclusive sample of hadronic events, $N_{\text{had}}^{\text{ch}}$, and then subtracts the contamination of heavy-quark-initiated events. This can be done if one knows the fractions of light- and heavy-quark events present in the sample, f_ℓ and f_Q, as well as the difference between the mean multiplicities of the heavy and light quarks, $\delta_{Q\ell}$, using the relation

$$N_{\text{had}}^{\text{ch}} = f_\ell \cdot N_{q\bar{q}}^{\text{ch}} + f_c \cdot (N_{q\bar{q}}^{\text{ch}} + \delta_{c\ell}) + f_b \cdot (N_{q\bar{q}}^{\text{ch}} + \delta_{b\ell}). \quad (B.1)$$

At $\sqrt{s} = 8$ GeV, where only the c-quark-initiated events are produced on top of the light-quark events, a direct measurement of $\delta_{c\ell}$ is not available and, thus, its value must be evaluated from the knowledge of experimentally measured values of $\delta_{c\ell}$ at different energies. Moreover, the knowledge of $\delta_{c\ell}$ is necessary to derive $\delta_{b\ell}$ from the results of those experiments which do not measure directly the c-quark event mean multiplicity. The measurement of $\delta_{c\ell}$ is difficult because it is not easy to select experimentally a highly enriched sample of c-quark-initiated events.

So far, only five experiments have published results on the direct measurement of the mean charged particle multiplicities, N_c^{ch} and N_ℓ^{ch}, for $e^+e^- \to c\bar{c}$ and $e^+e^- \to \ell\bar{\ell}$ ($\ell\bar{\ell} \equiv q\bar{q} = u\bar{u}, d\bar{d}, s\bar{s}$) events, including the evaluation of statistical and systematic uncertainties: MARKII [29] and TPC [32] at $\sqrt{s} = 29$ GeV, TASSO [33,34] at $\sqrt{s} = 35$ GeV and OPAL [41] and SLD [42,45] at $\sqrt{s} = 91.2$ GeV. These results, together with the derived values of $\delta_{c\ell}$ and their weighted average are presented in Table 1.[5]

It should be mentioned that the two results from SLD [42,45] were obtained from two completely independent event samples. The most recent one was collected with an upgraded detector, using a different experimental procedure and with different sources of systematic errors. We then consider the two results practically uncorrelated. It should also be noticed that the results of MARKII and TPC presented in Table 1 are different from those derived in [15], and used to evaluate the light-quark charged mean multiplicity at $\sqrt{s} = 8$ GeV in the same article. This is simply due to the fact that in [15] the values of N_ℓ^{ch} used to calculate $\delta_{c\ell}$ were not those quoted in the publications [29] and [32], but they were recalculated assuming a common mean value for the total average multiplicity, $N_{\text{had}}^{\text{ch}}$, as determined by different experiments at energies surrounding

[5] There are published results on the measurements of N_c^{ch} and N_ℓ^{ch} also at LEP2 energies [43,44,49]. Unfortunately, the limited statistics available at each energy did not allow for efficient c-quark tagging, comparable to that at the Z^0 peak. Therefore, the selection of highly enriched c-quark samples was not possible. As a consequence the measurements of N_c^{ch} are affected by large uncertainties, and cannot be used for a meaningful evaluation of $\delta_{c\ell}$.

Table 1. Mean charged particle multiplicities, $N_c^{\rm ch}$ and $N_\ell^{\rm ch}$, for $c\bar{c}$ and $\ell\bar{\ell}$ ($\ell\bar{\ell} = u\bar{u}, d\bar{d}, s\bar{s}$) events and the difference $\delta_{c\ell} = N_c^{\rm ch} - N_\ell^{\rm ch}$, measured at different energies. The results are corrected for detector effects as well as for initial state radiation effects. Charged decay products from K_S^o and Λ decays are included. We derived $N_\ell^{\rm ch}$ for TASSO from the published values of $N_b^{\rm ch}$, $N_c^{\rm ch}$ and $N_{\rm had}^{\rm ch}$, assuming the standard model quark fractions. The quoted errors are obtained by combining the statistical and the systematic errors in quadrature. OPAL and SLD errors on $\delta_{c\ell}$ were published considering also correlations. The weighted average assumes no correlations among the various experimental results

Experiment	$\sqrt{s}$ (GeV)	$N_c^{\rm ch}$	$N_\ell^{\rm ch}$	$\delta_{c\ell}$
MARKII [29]	29	13.2 ± 1.0	12.2 ± 1.4	1.0 ± 1.7
TPC [32]	29	13.5 ± 0.9	12.0 ± 0.9	1.5 ± 1.3
TASSO [33, 34]	35	15.0 ± 1.2	11.9 ± 1.2	3.1 ± 1.6
OPAL [41]	91.2	21.52 ± 0.62	20.82 ± 0.44	0.69 ± 0.62
SLD [42]	91.2	21.28 ± 0.61	20.21 ± 0.24	1.07 ± 0.59
SLD [45]	91.2	21.096 ± 0.653	20.048 ± 0.316	1.048 ± 0.718
Average				1.03 ± 0.34

Table 2. Corrected mean charged particle multiplicities and $\delta_{b\ell}$ at different energies (see text in Appendix B.2 for more details). According to [46], the DELCO result appearing in this table was corrected by +25% as compared to the published DELCO data, i.e. 3.6 ± 1.5, to account for the overestimated b purity of the selected sample

Experiment	$\sqrt{s}$	$N_{\rm had}^{\rm ch}$	$N_b^{\rm ch}$	$N_\ell^{\rm ch}$	$\delta_{b\ell}$
DELCO [30]	29	12.3 ± 0.8	15.2 ± 1.3	11.6 ± 0.9	4.5 ± 1.6
MARKII [29]	29	12.9 ± 0.6	16.1 ± 1.1	12.2 ± 1.4	3.9 ± 1.8
TPC [32]	29		16.7 ± 1.0	12.0 ± 0.9	4.7 ± 1.4
Average	29				4.4 ± 0.9
TASSO [33]	35	13.4 ± 0.66	16.0 ± 1.5	11.9 ± 1.2	4.1 ± 1.9
TASSO [34]	42.1	14.9 ± 0.7	17.0 ± 2.0	14.3 ± 1.5	2.7 ± 2.5
TOPAZ [35]	58	14.21 ± 0.12	16.24 ± 1.1	13.57 ± 0.7	2.7 ± 1.3
VENUS [36]	58	16.79 ± 0.23	19.38 ± 0.88	16.07 ± 0.7	3.31 ± 0.37
MARKII [37]	90.9	20.9 ± 0.5	23.1 ± 1.9	20.1 ± 0.9	3.0 ± 2.1
DELPHI [40]	91.2		23.32 ± 0.51	20.20 ± 0.45	3.12 ± 0.68
OPAL [41]	91.2		23.62 ± 0.48	20.82 ± 0.44	2.79 ± 0.30
SLD [42]	91.2		23.14 ± 0.39	20.21 ± 0.24	2.93 ± 0.33
SLD [45]	91.2		23.098 ± 0.378	20.048 ± 0.316	3.050 ± 0.311
OPAL [44]	130		25.9 ± 1.3	21.0 ± 1.4	4.9 ± 1.5
OPAL [44]	136		25.7 ± 1.7	23.0 ± 1.6	2.8 ± 2.0
OPAL [44]	161		24.1 ± 1.7	21.1 ± 2.1	2.9 ± 2.3
OPAL [44]	172		28.8 ± 2.2	26.8 ± 2.1	2.1 ± 2.5
DELPHI [43]	183		29.79 ± 1.14	25.25 ± 1.35	4.55 ± 1.5
OPAL [44]	183		28.3 ± 1.2	26.8 ± 1.6	1.5 ± 1.6
DELPHI [43]	189		30.53 ± 0.78	26.10 ± 0.97	4.43 ± 1.05
OPAL [44]	189		28.89 ± 0.77	25.41 ± 1.0	3.48 ± 1.2
OPAL [44]	192		28.5 ± 1.4	24.4 ± 1.9	4.1 ± 2.0
OPAL [44]	196		31.3 ± 1.5	28.6 ± 1.6	2.7 ± 1.7
DELPHI [43]	200		29.38 ± 0.82	25.99 ± 1.03	3.39 ± 1.35
OPAL [44]	200		30.3 ± 1.3	25.6 ± 1.5	4.7 ± 1.8
OPAL [44]	202		29.9 ± 1.7	25.5 ± 2.0	4.4 ± 2.0
DELPHI [49]	206		28.72 ± 0.77	24.22 ± 1.09	4.50 ± 1.17
OPAL [44]	206		30.08 ± 1.0	26.53 ± 1.4	3.55 ± 1.2

$\sqrt{s} \approx 29\,\text{GeV}$. This procedure was meant to reduce the uncertainty on the derived values of $\delta_{c\ell}$ and $\delta_{b\ell}$ but is rather dangerous since information about the strong correlations among $N^{\text{ch}}_{\text{had}}$, N^{ch}_{ℓ}, N^{ch}_{c} and N^{ch}_{b} existing within the same measurement is lost if one considers a mean value over different experiments for only one of these variables. That is why we rather preferred to use the published results which were all obtained within the same measurement.

There are two more experimental results on N^{ch}_{c} and N^{ch}_{ℓ} published in the literature, one by the HRS collaboration at $\sqrt{s} = 29\,\text{GeV}$ [31] and one by the DELPHI collaboration at $\sqrt{s} = 91\,\text{GeV}$ [40]. Unfortunately, only the statistical uncertainties were evaluated in these analyses, and since the contribution of the systematic errors to the total error quoted by the other experiments is important, or even dominant, we did not consider the results from HRS and DELPHI in our weighted average. We show in the following that in any case, under reasonable assumptions about the size of the total errors, the final result would not change significantly if we did. The DELPHI experiment measured N^{ch}_{c}, N^{ch}_{b} and N^{ch}_{ℓ} and found $\delta_{c\ell} = 1.64$. The total uncertainty (statistics and systematics combined) on $\delta_{b\ell}$ quoted in their analysis is about a factor two larger than those quoted by SLD [42, 45] and OPAL [41], and if we assume a similar relative precision also for $\delta_{c\ell}$, by comparison with the SLD and OPAL total uncertainties we get for DELPHI $\delta_{c\ell} = 1.64 \pm 1.2$. Our weighted average in Table 1 would change to $\langle\delta_{c\ell}\rangle = 1.07 \pm 0.33$ if we would include also this result.

The HRS experiment measured N^{ch}_{c} and N^{ch}_{ℓ}, and found $\delta_{c\ell} = 1.6$. The results on N^{ch}_{c} and N^{ch}_{ℓ} are consistent with those found by MARK II and TPC, and the size of the statistical errors are similar. If we attribute to the HRS value of $\delta_{c\ell}$ a total uncertainty similar to those quoted by MARKII and TPC (here we assume a total error of ± 1.5) and include also this measurement in our weighted average, we would get $\langle\delta_{c\ell}\rangle = 1.09 \pm 0.32$.

In conclusion, we use $\langle\delta_{c\ell}\rangle = 1.0 \pm 0.4$ in the present analysis, and we point out that considering the current experimental precision there is no evidence of energy dependence of $\delta_{c\ell}$ in the range $29\,\text{GeV} \leq \sqrt{s} \leq 91\,\text{GeV}$.

B.2 About the measurement of $\delta_{b\ell}$

In Table 2 we present an updated review of the experimental measurements of the mean charged particle multiplicities, $N^{\text{ch}}_{\text{had}}$, N^{ch}_{b} and N^{ch}_{ℓ}, respectively for the inclusive sample (when measured), $b\bar{b}$ events and $\ell\bar{\ell}$ ($\ell\bar{\ell} = u\bar{u}, d\bar{d}, s\bar{s}$) events. The difference $\delta_{b\ell} = N^{\text{ch}}_{b} - N^{\text{ch}}_{\ell}$ is also shown. The results are corrected for detector effects as well as for initial state radiation effects. Charged decay products from the K^{o}_{S} and Λ decays are included. The quoted errors are obtained by combining the statistical and the systematic uncertainties in quadrature. The published results on $\delta_{b\ell}$ from OPAL, SLD, DELPHI and VENUS take correlations into account. According to [46], the DELCO result appearing in Table 2 was corrected by +25% as compared to the published DELCO data, i.e. 3.6 ± 1.5, to account for the overestimated b purity of the selected sample.

We would like to stress at this point that the results on $\delta_{b\ell}$ presented in published compilations, including this one, are not all direct measurements. MARKII and TPC at $\sqrt{s} = 29\,\text{GeV}$, TASSO at 35 GeV, OPAL, SLD and DELPHI at 91 GeV and DELPHI and OPAL at LEP2 energies, measured N^{ch}_{b}, N^{ch}_{c} and either $N^{\text{ch}}_{\text{had}}$, the inclusive mean charged multiplicity, or N^{ch}_{ℓ} (or both), from which $\delta_{b\ell}$ is calculated in a direct way. The other experiments, instead, have only measured N^{ch}_{b} and $N^{\text{ch}}_{\text{had}}$, and, thus, one particular value for N^{ch}_{c} or $\delta_{c\ell}$ must be assumed in order to evaluate N^{ch}_{ℓ} and $\delta_{b\ell}$. In the previous reviews, the value of $\delta_{c\ell}$ was the same as in [15], while in the recent publication by VENUS [36] the result of OPAL measurement [38] is taken. In Table 2 we used for all these experiments the new average value of $\delta_{c\ell}$ presented in the previous section, $\delta_{c\ell} = 1.0 \pm 0.4$, and this explains why these results are not the same as those presented in previous publications.

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